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Curvature Estimation for Unstructured Triangulations of Surfaces

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Abstract

In this work, a survey of several curvature estimation methods for surface meshes was conducted, and a comparison of two types of curvature estimation techniques was conducted based on convergence studies. As a result of this work, a new and improved method was proposed as an extension of one of the surveyed methods. The new method robustly estimates normals, principal curvatures, mean curvatures and Gaussian curvatures at vertices of general unstructured triangulations. The method has been tested on complex meshes and has provided good results.

1 Introduction

Knowledge of the curvature of surfaces is important in a number of applications such as flow simulations, computer graphics and animations, and pattern matching. It is of particular importance to applications dealing with evolving surface geometry. Such applications usually do not have smooth analytical forms for the surfaces forming the model geometry. Instead, they have to deal with discrete data consisting of points on the surface connected to form an unstructured mesh. Hence, it is important to be able to reliably estimate local curvatures at points on such discrete surfaces.

In this work, a survey of several curvature estimation methods [1, 2, 3, 4, 5] for surface meshes was conducted, and a comparison of two types of curvature estimation techniques was conducted based on convergence studies. As a result of this work a new and improved method was proposed as an extension of one of the surveyed methods. The new method robustly estimates normals, principal curvatures, mean curvatures and Gaussian curvatures at vertices of general unstructured triangulations. The method has been tested on complex meshes and has provided good results.

There are three classes of methods for estimating higher order information such as curvature from discrete surface information (e.g. nodes of a triangulation). The first class of methods builds a global or local parametrization for the discrete surface and computes curvature information from the derivatives of the parametrization [2, 6, 5]. The second class of methods fits a smooth surface to a local set of points around the point of interest and uses the curvature of the surface at that point as the curvature estimate. These methods do not necessarily need the points to be connected to form a mesh in order to estimate the curvature, although initial estimation of the surface normal is facilitated by the availability of such a mesh. The smooth surface is usually chosen to be a limited form of a quadratic polynomials [2]. The third class of methods estimates the curvature directly from the triangulation by a variety of methods [3, 4] such as discrete differential geometry.

In this work, the discrete differential geometry approach and 2 different types of quadric fitting approaches were studied, and a convergence study conducted to see which method provided the most accurate results. In the following sections, the methods are first described briefly. Then the test setup for the convergence study is described, followed by the results of the tests. Finally, results on large meshes are presented to show that the chosen method works well for complex triangulations.

2 Curvature Estimation using Discrete Differential Geometry Operations

In Meyer et.al. [3], a curvature estimation method using spatial averaging of triangulation data using discrete differential geometry concepts is outlined. The Gaussian curvature is estimated using a discretized form of the Gauss-Bonet theorem applied in the 1-ring neighborhood of a vertex, i.e., over the set of triangles connected to the vertex. The mean curvature estimate is derived from a discretization of the Laplace-Beltrami operator also applied to the 1-ring neighborhood. Given a patch of triangles surrounding point $\mathbf{x}_i$ as shown in Figure 1, the estimates for the Gaussian curvature, K_i and mean curvature H_i, at $\mathbf{x}_i$, given by Meyer et.al. are:

$$K_i = \frac{1}{A}\left(2\pi - \sum_j \theta_j\right) \tag{1}$$

$$2H_i\hat{\mathbf{n}}_i = \frac{1}{2A}\sum_j (\cot\alpha_{ij} + \cot\beta_{ij})(\mathbf{x}_i - \mathbf{x}_j) \tag{2}$$

where A is some area around $\mathbf{x}_i$, $\hat{\mathbf{n}}_i$ is the normal vector at $\mathbf{x}_i$ defined by Eq. 2, and where $\mathbf{x}_j$, θ_j, α_{ij} and β_{ij} are as shown in the figure. Meyer et.al. show that the error in the curvature computation is minimized when A is chosen to be the "Voronoi area", defined in each triangle by the point $\mathbf{x}_i$, the midpoints of the triangle edges, and the circumcenter of the triangle, summed over all the triangles. The Voronoi area suggested by the authors is given by:

$$A_{vor} = \frac{1}{8}\sum_j (\cot\alpha_{ij} + \cot\beta_{ij})\|\mathbf{x}_i - \mathbf{x}_j\|^2 \tag{3}$$

In the case of the obtuse triangles (where the circumcenter is outside the triangle), they suggest using a modified area using the midpoint of the edge opposite to the obtuse angle instead of the circumcenter with the consequence that the errors are higher. Although the authors show that the error is minimized by using Voronoi areas, they do not present any convergence results.

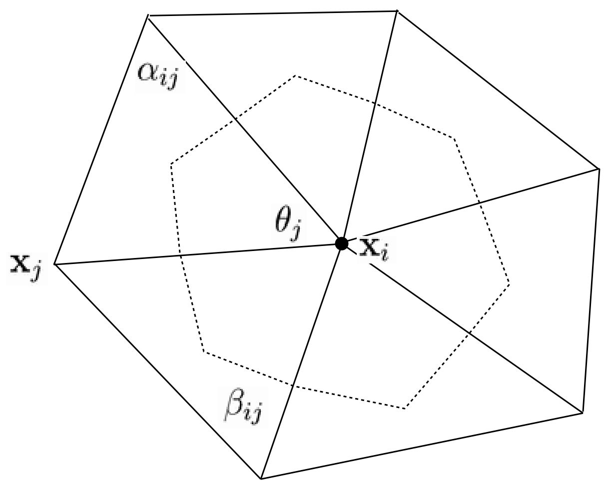


Figure 1: 1-ring neighborhood of vertex indicating the sub-area used for computation using the method of Meyer et.al.

3 Curvature estimation by fitting quadrics

A survey of quadric fitting methods for estimating surface properties such as curvature is presented by McIvor and Valkenburg [2] and more recently by Petitjean [7]. Quadric fitting methods are based on the idea that smooth surface geometry can be locally approximated using a quadratic polynomial surface. Therefore, quadric fitting methods try to fit a quadric to points in a local neighborhood of each point of interest. The quadric is fitted in a local coordinate frame positioned at the point of interest and with the Z-coordinate axis aligned along an estimate for the surface normal at that point. Then, the curvature of the quadric at the point of interest is taken to be the estimate of the curvature for the discrete surface.

The detailed procedure for fitting a simple quadric of the form $Z' = aX'^2 + bX'Y' + cY'^2$ is given step-by-step [7] as:

1. Estimate the surface normal $\hat{\mathbf{n}}$ at the point of interest $\mathbf{p}$. This estimation can be done by simple or weighted averaging of the neighboring triangle normals or by finding a least squares fitted plane to the point and its neighbors.

2. Position a local coordinate system (X', Y', Z') at the point with the Z' coordinate along the estimated normal. To fix the X' coordinate axis, McIvor and Valkenburg suggest aligning it with the projection of the global X axis onto the tangent plane defined by $\hat{\mathbf{n}}$. This results in a rotation matrix, $\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]^T$, from the global coordinate frame to the local coordinate frame, where:

$$\mathbf{r}_3 = \hat{\mathbf{n}} \qquad \mathbf{r}_1 = \frac{(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)\hat{\mathbf{i}}}{\|\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T\|\hat{\mathbf{i}}} \qquad \mathbf{r}_2 = \mathbf{r}_3 \times \mathbf{r}_1 \tag{4}$$

in which $\mathbf{I}$ is the identity matrix and $\hat{\mathbf{i}}$ is the global X axis $[1, 0, 0]^T$.

A situation not discussed in [2] and [7] is the degenerate case when the normal is aligned with the global X-axis and $\mathbf{r}_1$ computed above is the null vector. In such a case, one can align the X' axis with the projection of the global Y axis on the tangent plane by replacing $\hat{\mathbf{i}}$ with $\hat{\mathbf{j}}$ in the equation for $\mathbf{r}_1$.

3. Select the set of points to be used in fitting the quadric. The simplest choice the set of edge-connected neighbors of the node under consideration.

4. Map the coordinates of the selected points from the global to the local coordinate system using the relation $\mathbf{x}' = \mathbf{R}(\mathbf{x} - \mathbf{p})$.

5. Solve for the coefficients of the quadric by computing a least squares solution of the equations

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n y_n & y_n^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \tag{5}$$

   Note: the least squares solution to an overdetermined system $\mathbf{Ax} = \mathbf{b}$ such Eq. 5 is given by $\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.

6. Then the estimate of principal curvatures, κ_1 and κ_2, the Gaussian curvature K and the mean curvature H, all at $\mathbf{p}$, are given by:

$$\kappa_1 = a + c + \sqrt{(a-c)^2 + b^2} \tag{6}$$

$$\kappa_2 = a + c - \sqrt{(a-c)^2 + b^2} \tag{7}$$

$$K = 4ac - b^2 \tag{8}$$

$$H = a + c \tag{9}$$

Estimates of the curvature using suvh a simple quadric are quite sensitive to estimates of the surface normal. The *extended quadric* fitting attempts to reduce this sensitivity by including linear terms in the quadric that is fitted, i.e., the quadric is given by $Z' = aX'^2 + bX'Y' + cY'^2 + dX' + fY'$. The coefficients of such a quadric can then be used to compute a new estimate for the surface normal as:

$$\hat{\mathbf{n}} = \frac{1}{(d^2 + e^2 + 1)^{1/2}}[-d, -e, 1]^T \tag{10}$$

Using this new estimate of the surface normal, a new local coordinate system can be computed at the point of interest, and a new quadric can be fitted to the points. The process can be repeated until the surface normal estimates converge, and the coefficients of the resulting quadric can be used to calculate the curvatures as:

$$K = \frac{4ac - b^2}{(1 + d^2 + e^2)^2} \tag{11}$$

$$H = \frac{a + c + ae^2 + cd^2 - bde}{(1 + d^2 + e^2)^{3/2}} \tag{12}$$

It is also possible to extend the method by fitting quadrics with a non-zero constant term to the points in a local neighborhood of the point of interest. Such a method, called the *full quadric method*, allows the quadric to not pass through the point of interest; however, this approach was not considered here.

One disadvantage of the extended quadric method is that it requires at least five points to obtain a unique solution instead of the three points required by the simple quadric method. However, the method has been adapted in this study so that the local neighborhood of a point is extended to its 2-level neighbors if it does not have enough 1-level neighbors. This allows for robust computation of the curvature at all points of the mesh that are not on the surface boundary. On the surface boundary, the method has been extended by reflecting the neighbors of the point to form ghost points so that the curvature estimate is not one-sided.

The improved method that corrects for the degenerate case of the rotation matrix, for the lack of sufficient points in the neighborhood and for the one-sidedness of the curvature estimate on the boundary is named the extended quadric, extended patch method. The extended quadric, extended patch method allows accurate and robust curvature computations for complex unstructured triangulations.

4 Convergence Tests

Consider a hexagonal patch of triangles of radius L, whose vertices $\mathbf{x}_i,\ i = 1, 6$ and center $\mathbf{p}$ all lie on a cylindrical surface $\mathcal{C}$ of fixed radius $r < L$[1]. Assume that axis of the cylinder coincides with Z-axis and that the X-axis passes through the central vertex. Then the coordinates of the central vertex $\mathbf{p}$ are given by $(r, 0, 0)$. The coordinates of $\mathbf{x}_2$ and $\mathbf{x}_5$ in the patch are $(r, 0, -L)$ and $(r, 0, L)$ respectively, putting them directly below and above vertex $\mathbf{p}$ in the axial direction.

If the coordinates of an outer node of the hexagonal patch is given by (x_i, y_i, z_i), then the following relations must hold true

$$x_i^2 + y_i^2 = r^2 \qquad \text{from the equation of the cylinder} \tag{13}$$

$$(x_i - r)^2 + y_i^2 + z_i^2 = L^2 \qquad \text{length of the hexagon's "spokes" are all } L \tag{14}$$

Assuming z_i to be at $\pm L/k_i$, we get

$$x_i = \frac{1}{2r}\left[r^2 - L^2\left(1 - \frac{1}{k_i^2}\right)\right] \tag{15}$$

which can then be substituted in Eq. 13 to calculate y_i.

For the hexagonal patch to be regular, it can be seen that $z_4 = z_6 = L/2$ and $z_1 = z_3 = -L/2$. Therefore, for a regular hexagon, one can write

$$x_i = \frac{1}{2r}\left[r^2 - \frac{3}{4}L^2\right] \tag{16}$$

Since this study is directed towards arbitrary unstructured meshes, it is more useful to study an irregular hexagonal patch of triangles. To accomplish this, one of the vertices, namely $\mathbf{x}_3$, of the regular hexagon is moved to so that $z_3 = -4L/5$. Then the remaining two coordinates are computed above equations so that the point remains on

[1] More specifically, each vector $\mathbf{x}_j - \mathbf{p}$ lies in its own half-plane, the half-planes have the line N normal to $\mathcal{C}$ at $\mathbf{p}$ as their common line of intersection, and each $\mathbf{x}_j$ also lies on a sphere of radius L centered on $\mathbf{p}$.

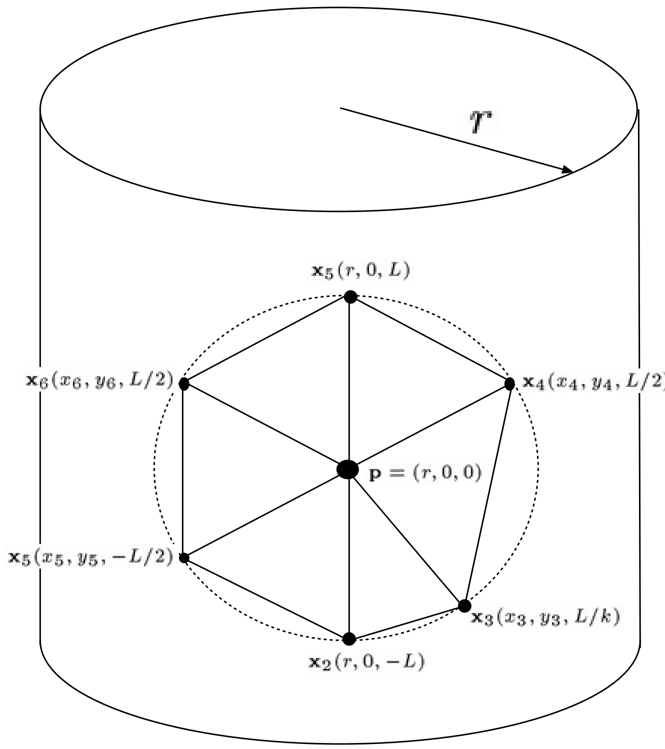


Figure 2: Hexagonal patch of elements on a cylinder

the cylinder and the sphere circumscribing the hexagon. This has the effect of only altering the angles of two triangles in the patch and leaving the characteristic mesh size (defined by the length, L, of the internal edges or "spokes" of the hexagon) unchanged.

Using the above formulas, one can reduce L, keeping r fixed, in order to reduce the size of the hexagon. As the hexagon gets smaller, the nodes of the hexagon continue to be on the cylinder, the quality of the triangles does not change much and the nodes become more coplanar.

The convergence of any curvature estimation method can then be checked, by reducing L from some intial value to some final value and estimating the curvature at each step. In this work, the radius of the cylinder r is taken to be 2.0 and the length of each "spoke" of the hexagon, L, is decreased from 0.5 to 0.05 in steps of 0.05. The results of the tests with the spatial averaging method of Meyer et.al., the simple quadric method, and the extended quadric method are tabulated in Table 1 and shown in Figure 3.

One can see from the results that the spatial averaging method of Meyer et.al. has bounded error but does not converge; moreover, this error is the highest of the three methods tested (≈7%). The method of fitting simple quadrics converges to an incorrect value that has 0.74% error. Elementary analysis of this method shows that if the normal vector used is first order accurate, then the simple quadric fit yields zero'th order accuracy (constant error) for the associated curvature estimates. The extended quadric performs the best among the three methods, yielding curvatures converging to the correct value. Again, an associated analysis shows that, here, the surface normal converges quadratically (cubically for some surfaces) while the curvature exhibits linear

Mesh size (L)	Spatial Avg. of [3]	Simple Quadric	Ext. Quadric
0.5000	0.231580	0.250790	0.252723
0.4500	0.231583	0.250287	0.252204
0.4000	0.231475	0.249711	0.251752
0.3500	0.231590	0.249447	0.251336
0.3000	0.231577	0.249092	0.250970
0.2500	0.231583	0.248810	0.250678
0.2000	0.231572	0.248559	0.250420
0.1500	0.231570	0.248376	0.250230
0.1000	0.231640	0.248323	0.250172
0.0500	0.231654	0.248257	0.250107
0.0050	0.231596	0.248168	0.250024
0.0005	0.231575	0.248145	0.250000

Table 1: Mean curvature estimates for varying size of hexagonal patch of elements on a cylinder using three different methods

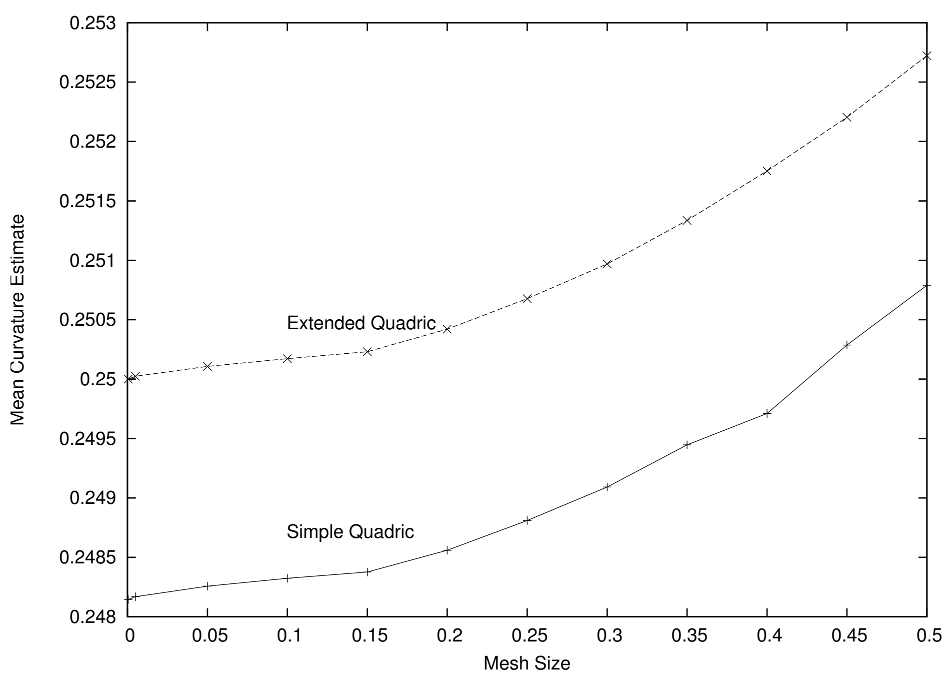


Figure 3: Convergence of Quadric Fitting Methods for Curvature Estimation on Hexagonal Patch of Elements

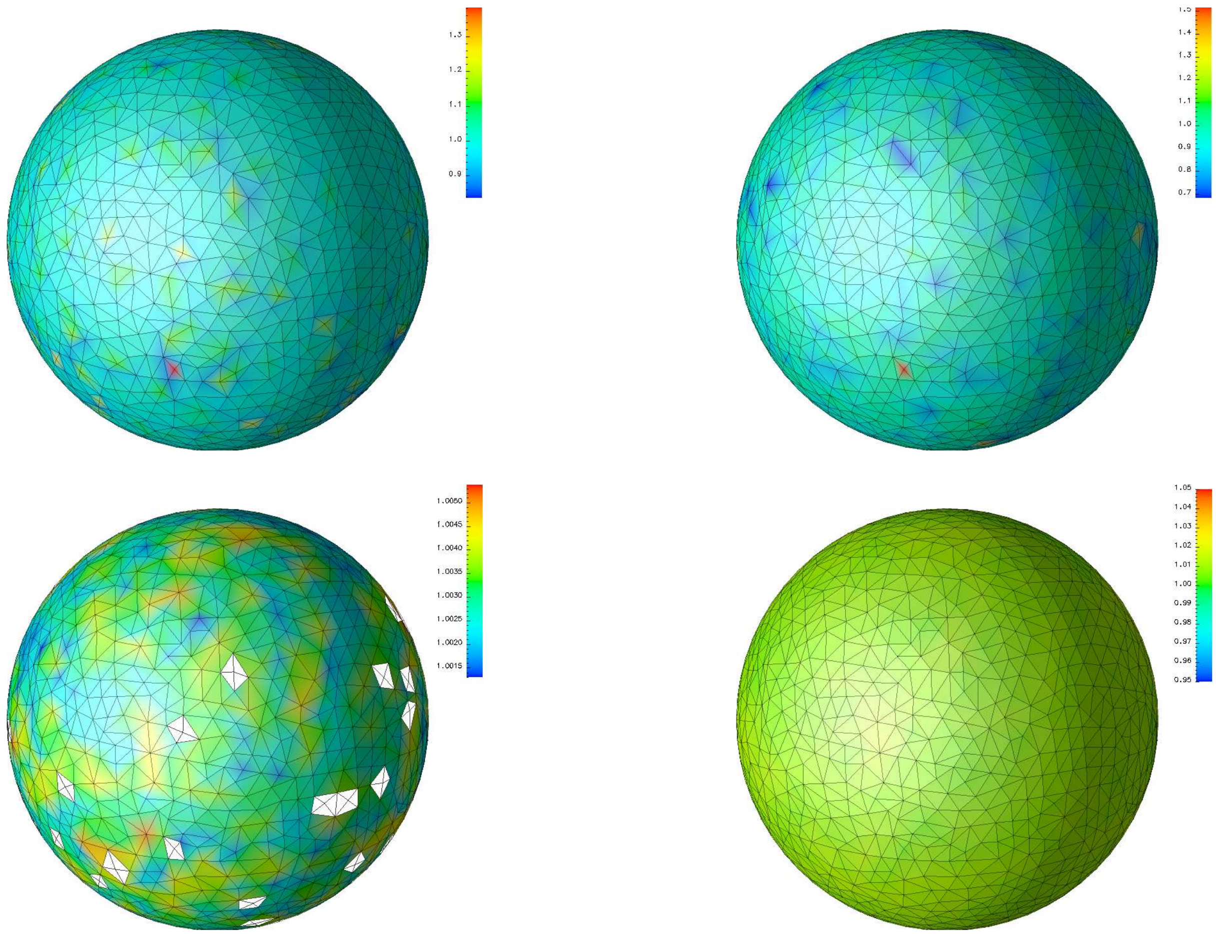


Figure 4: Mean curvature estimates on unstructured mesh of sphere by (a) Spatial Averaging Method of Meyer et.al. (b) Simple Quadric Fitting (c) Extended Quadric Fitting (white patches indicate locations at which an extended quadric could not be fitted due to insufficient points) (d) Extended Quadric Fitting on an extended patch

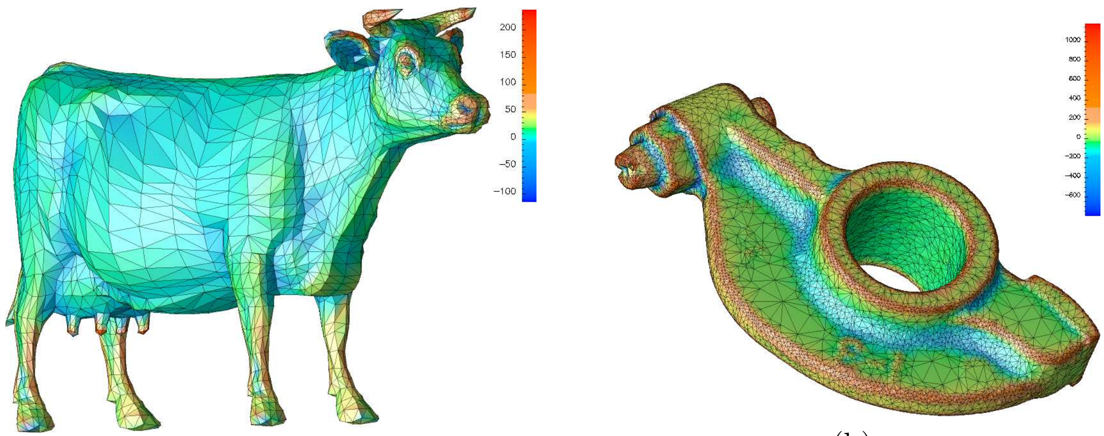
convergence.

5 Results on General Meshes

The modified curvature estimation method was tried on several complex unstructured meshes and has given good results.

Figure 4 shows the mean curvature distribution on the unstructured mesh of a sphere of radius 1.0 computed by the various methods. As seen in the figures, the results of the spatial averaging method of Meyer et.al. and the simple quadric method are not very good. The extended quadric method gives excellent results (except that it is not defined at locations where a node is connected to less than five adjacent nodes). The extended quadric, extended patch method solves this problem and accurately estimates the curvature at all points with a maximum error of 1.6%.

Figure 5a, Figure 5b and Figure 6 show the mean curvature distribution on unstruc-



(a) (b)

Figure 5: Mean curvature estimates for cow and rocker arm (mesh courtesy of Cyberware, Inc.)

tured meshes of a cow, a rocker arm and an archaelogical artifact calculated using the extended quadric, extended patch method. As seen from these figures, the method estimates qualitatively reasonable curvatures at various features of these complex meshes.

6 Conclusions

In this study several curvature estimation methods for unstructured meshes were surveyed, and three methods were studied more closely to see which provided the best results. It was concluded that the extended quadric method with extensions devised by the authors was the most accurate and robust procedure for estimating local curvatures on complex surface triangulations.

7 Acknowledgements

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References

[1] B. Hamann. Curvature approximation for triangulated surfaces. *Computing Suppl.*, 8:139–153, 1993.

[2] A. M. McIvor and R. J. Valkenburg. A comparison of local surface geometry estimation methods. *Machine Vision and Applications*, 10:17–26, 1997.

[3] M. Meyer, H. Lee, M. Desbrun, and A. H. Barr. Generalizing barycentric coordinates for irregular n-gons. *Journal of Graphics Tools*, 2002. to appear.

[4] G. Taubin. Estimating the tensor of curvature of a surface from a polyhedral approximation. In *Proceedings of the Fifth International Conference on Computer Vision (ICCV 95)*, pages 852–857, 1995.

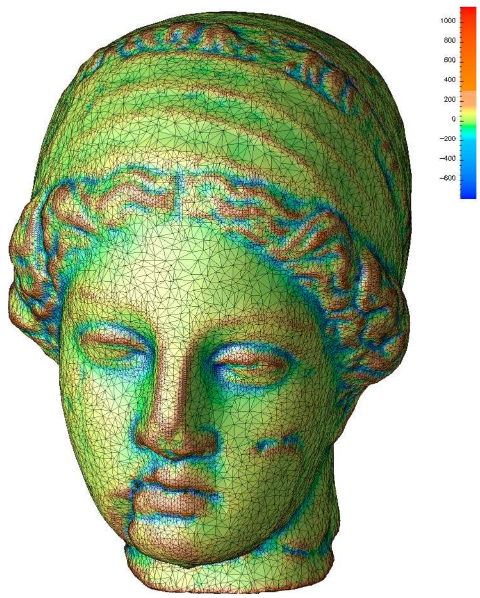

Figure 6: Mean curvature estimates for Igea artifact (mesh courtesy of Cyberware, Inc.)

[5] F. Mokhtarian, N. Khalili, and P. Yuen. Curvature computation on free-form 3-d meshes at multiple scales. *Computer Vision and Image Understanding*, 83(2):118–139, 2001.

[6] R. M. Haralick and L. G. Shapiro. *Computer and Robot Vision*, volume I, chapter 8. Addison-Wesley, 1992.

[7] S. Petitjean. A survey of methods for recovering quadrics in triangle meshes. *ACM Computing Surveys*, 34(2):211–262, June 2002.